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ITERATED FORCING IN QUADRATIC FORM THEORY*

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ABSTRACT

In [Sp1] and [B/Sp] it has been shown that the existence of quadratic spaces of uncountable dimension over finite or countable fields sharing the property that every infinite dimensional subspace has its orthogonal complement of at most countable dimension is independent of the axioms of ZFC set theory. Such a space will be called a *strong Gross* space in the sequel. Cardinal invariants of the continuum decide whether strong Gross spaces exist or not. Namely, when $\mathbf{b} = \omega_1$ a strong Gross space of dimension $\aleph_1$ exists. When $\mathbf{p} > \omega_1$ such spaces do not exist. Here we answer the question what happens with strong Gross spaces in case $\mathbf{b} > \omega_1$ or $\mathbf{p} = \omega_1$.

0. Introduction

In [Ba/G], a symmetric bilinear space (E, Φ) of uncountable dimension has been constructed over an arbitrary finite or countable field, sharing the property

(**) for all subspaces $U \subseteq E$: if $\dim U \geq \aleph_0$ then $\dim U^\perp \leq \aleph_0$

The construction could be done only under the assumption that the Continuum Hypothesis (CH) holds in the underlying set theory.

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The (**)-property arose from investigations of the orthogonal group of quadratic spaces in [G/O]. In [G/O] it has been shown that (**)-spaces of uncountable dimension always (in ZFC) exist provided the base field is uncountable. Note that a space of countable or finite dimension is trivially (**). In the sequel, a quadratic space will be called **strongly Gross** if it has the (**)-property, its dimension is uncountable and its base field is countable or finite.

In [Sp1] and [B/Sp] it has been shown that the existence of strong Gross spaces is independent of the axioms of ZFC set theory and, conversely, that strong Gross spaces can exist in models where CH fails. Involved are cardinal invariants of the continuum, especially $\mathbf{b}$ and $\mathbf{p}$. Here $\mathbf{b}$ is the minimal cardinality of a family of functions $\omega \to \omega$ which is unbounded with respect to the quasi-ordering $<^*$ defined by $f <^* g$ if $f(n) < g(n)$ for all but finitely many n, whereas $\mathbf{p}$ is defined as the minimal cardinality of a filter on the natural numbers such that there exists no infinite set which is almost included in every member of the filter. For $\mathbf{b}$ and $\mathbf{p}$ the relations $\omega_1 \leq \mathbf{p} \leq \mathbf{b} \leq \mathbf{c}$ are provable in ZFC. (By $\mathbf{c}$ we denote the cardinality of the continuum 2^ω.) In ZFC, none of these relations can be proved to be either an equality or a strict inequality.

In models for ZFC where $\mathbf{p} > \omega_1$ holds strong Gross spaces do not exist. Martin's Axiom (MA) implies $\mathbf{p} > \omega_1$. Conversely, if $\mathbf{b} = \omega_1$ holds, then a strong Gross space of dimension $\aleph_1$ exists over every field which is the extension of an arbitrary finite or countable field by countably many transcendentals ([B/Sp]). In [Sh/Sp] this result has been generalized to arbitrary infinite fields. However, $\mathbf{b} = \omega_1$ does not imply that a strong Gross space exists over a finite field (see [Sh/Sp]).

The obvious questions remaining open are:

Question 1: Is the assumption $\mathbf{b} > \omega_1$ (weaker than $\mathbf{p} > \omega_1$) strong enough to prove that strong Gross spaces do not exist?

Question 2: Is the assumption $\mathbf{p} = \omega_1$ (weaker than $\mathbf{b} = \omega_1$) strong enough to prove that strong Gross spaces exist?

In other words: Is one of the statements "$\mathbf{b} = \omega_1$" or "$\mathbf{p} = \omega_1$" equivalent to "Strong Gross spaces exist"?

Here we give negative answers to both questions.

In Chapter 2 we show that strong Gross spaces are not destroyed by forcing with the natural partial ordering to adjoin a dominating function (Hechler forcing). Iterating this forcing over a model where strong Gross spaces exist we get a model where such spaces still exist and $\mathbf{b} > \omega_1$ holds.

In Chapter 3 we define a partial ordering P^k with the ccc (countable chain condition), where k is an arbitrary finite or countable field, such that no quadratic space over k which is in the ground model is strongly Gross in the extension V^{P^k} Iterating this forcing such that all possible fields are taken into consideration we arrive at a model where no strong Gross spaces exist. By choosing the ground model appropriately we make sure that $\mathbf{p} = \omega_1$ holds in the extension.

In order to motivate the word "strong" in "strong Gross space" we mention that the notion of "Gross space" is obtained by replacing in the definition of a strong Gross space the (∗∗)-property by the following weaker property

(∗) for all subspaces $U \subseteq E$: if $\dim U \geq \aleph_0$ then $\dim U^{\perp} < \dim E$

In [B/Sp] and [Sh/Sp] there are partial independence results concerning Gross spaces. Gross spaces of dimension $\mathbf{c}$ exist if MA holds. Note that this contrasts with the situation for strong Gross spaces. In the model obtained by iterated perfect set forcing there are no Gross spaces in dimension $\mathbf{c}$. The partial ordering P^k from Chapter 3 can also be used to produce such a model. But it is still an open question whether there is a ZFC-model where Gross spaces do not exist in any dimension.

For a survey on the whole subject see [Sp4].

1. Preliminaries

1.1. The cardinal invariants b and p.

If f, $g \in {}^{\omega}\omega$ then we say f **eventually dominates** g, and we write $g <^* f$, iff $\exists k \forall n \geq k\, g(n) < f(n)$. A family $F \subseteq {}^{\omega}\omega$ is **$<^*$-unbounded** iff there is no $f \in {}^{\omega}\omega$ such that $\forall g \in F\, g <^* f$. Then $\mathbf{b}$ is defined as the minimal cardinality of a $<^*$-unbounded family in ${}^{\omega}\omega$.

Let $[A]^{\omega}$ denote the set of all countably infinite subsets of the set A. For a, $b \in [\omega]^{\omega}$, let $a \subseteq^* b$ iff $a - b$ is finite. If $\mathcal{F} \subseteq [\omega]^{\omega}$ then a set $a \subseteq \omega$ is called **pseudo-intersection** of $\mathcal{F}$ iff $\forall b \in \mathcal{F}\, a \subseteq^* b$. We say that $\mathcal{F}$ has the **strong finite intersection property** iff every finite set of elements of $\mathcal{F}$ has infinite intersection. Then $\mathbf{p}$ is defined as the minimal cardinality of a family in $[\omega]^{\omega}$ with the strong finite intersection property which has no infinite pseudo-intersection.

We quote the following theorems from [vD] which form part of the folklore in set theory.

THEOREM 1 (part of Theorem 3.1.): a) $\omega_1 \le \mathbf{p} \le \mathbf{b} \le \mathbf{c}$.

b) $\mathbf{b}$ *and* $\mathbf{p}$ *are regular.*

c) *If* $\omega \le \kappa < \mathbf{p}$ *then* $2^\kappa = \mathbf{c}$.

THEOREM 2 (part of Theorem 5.1.): *Let* κ *and* λ *be regular cardinals with* $\omega_1 \le \kappa \le \lambda$. *It is consistent with* ZFC *that* $\mathbf{c} = \lambda$ *and* $\mathbf{b} = \mathbf{p} = \kappa$.

THEOREM 3 (a consequence of Theorems 5.3 and 3.1.): *It is consistent with* ZFC *that* $\mathbf{b} > \mathbf{p}$.

Remark: In this paper, two models of ZFC are investigated where $\mathbf{b} > \mathbf{p}$ holds. ∎

1.2. ITERATED FORCING.

On iterated forcing we agree with [B], [J] or [K]. We consider only finite-support iterations. Thus an iteration looks like $\langle P_\beta : \beta \le \alpha \rangle$, where P_β is the result of the first β stages of the iteration (the direct limit if β is a limit ordinal), and for each $\beta < \alpha$ there is some P_β-name $\dot{Q}_\beta$ such that

$$\Vdash_{P_\beta} \dot{Q}_\beta \text{ is a partial ordering}$$

and $P_{\beta+1}$ is isomorphic to the forcing product $P_\beta * \dot{Q}_\beta$. Throughout this paper we use the following well-known facts about iterated forcing (sometimes without explicitly mentioning them).

THEOREM 4 ([B], Theorem 1.2.): *If* $\beta < \alpha$, G_α *is* P_α*-generic over* V *and* $G_\beta = \{p|\beta : p \in G_\alpha\}$, *then* G_β *is* P_β*-generic over* V. *Thus we have* $V[G_\beta] \subseteq V[G_\alpha]$.

THEOREM 5 (essentially [B], Lemma 3.5; see also [K], Lemma 5.14, p. 276.): *Suppose* P_α *has the countable chain condition. If* G_α *is* P_α*-generic over* V *and in* $V[G_\alpha]$, $X \subset V$ *is a set of cardinality* $< cf(\alpha)$, *then there exists* $\beta < \alpha$ *such that* $X \in V[G_\beta]$.

THEOREM 6 ([B], Cor. 2.3.): *Assume that for every* $\beta < \alpha \Vdash_{P_\beta} \dot{Q}_\beta$ has the countable chain condition. *If direct limits are taken everywhere then* P_α *has the countable chain condition.*

THEOREM 7 ([B], Lemma 3.2 in case $\lambda = \omega_1$): *Suppose P has the countable chain condition, $|P| \leq \kappa$ and $\kappa^\omega = \kappa$. If $\Vdash_P |\dot{Q}| \leq \kappa$, then $|P * \dot{Q}| \leq \kappa$.*

THEOREM 8 ([B], Lemma 3.3 in case $\lambda = \omega_1$): *Suppose P has the countable chain condition, $|P| \leq \kappa$ and $\kappa^\omega = \kappa$. Then $\Vdash_P \mathbf{c} \leq \kappa$.*

1.3. FORMS.

We consider vector spaces E over a countable or finite commutative field k of arbitrary characteristic which are equipped with a symmetric bilinear form $\Phi : E \times E \to k$. For a subspace $U \subseteq E$ the **orthogonal** complement $U^\perp$ is defined as $U^\perp = \{x \in E : \forall y \in U \ \Phi(x,y) = 0\}$. The space (E, Φ) is called **nondegenerate** if $E^\perp = \{0\}$. Needless to say that a nondegenerate space may contain **isotropic** vectors, i.e. nonzero vectors x such that $\Phi(x,x) = 0$.

We remark that all the results in this paper and in [B/Sp] are true in a fairly more general context. At first, everything remains true if we generalize to orthosymmetric sesquilinear forms over a skew field which is endowed with an involutory antiautomorphism (see [G] for the definitions), since what is needed essentially is that the form defines a symmetric orthogonality relation. Moreover, using a representation theorem (for AC-lattices of length ≥ 3 equipped with a polarity (see [G] and [Ma/Ma])), then all the results can be transferred to the level of abstract ortho-lattices. Of course, it should as well be possible to carry out all the proofs on this level. This has been announced in [Sp2] and is worked out in [Sp3].

2. Existence of strong Gross spaces is consistent with $\mathbf{b} > \omega_1$

By Theorem 1 in [Ba/G], strong Gross spaces of dimension $\aleph_1$ exist under the assumption that CH holds. In [B/Sp] it has been shown that CH is not necessary for this result. In fact, forcing with the Cohen algebra to enlarge the continuum produces strong Gross spaces. We quote Theorem 2, Chapter 2, which cannot be improved since $\mathbf{c}$ is an upper bound for the dimension of a strong Gross space. Since k is countable or finite this follows quickly from the Erdös-Rado partition Theorem. More general, it is not difficult to see that a space (E, Φ) which is defined over a field k of arbitrary cardinality and has the $(**)$-property satisfies $\dim E \leq |k|^\omega$.

THEOREM 1 ([B/Sp], Theorem 2, Ch. 2.): *Suppose we force by adding κ Cohen reals where $\kappa > \aleph_0$. Let E be a κ-dimensional vector space over a finite or* *countable field k in the extension.*

Then there exists a nondegenerate symmetric bilinear form Φ on E such that for any countable linearly independent set $X \subseteq E$, for all but countably many vectors $y \in E$ it is true that $\{\Phi(x,y) : x \in X\} = k$. Hence, in particular, (E, Φ) is strongly Gross.

In [B/Sp] (Chapter 4) a strong Gross space of dimension $\aleph_1$ and over a countable field has been constructed under the sole condition $\mathbf{b} = \omega_1$. The question remained open whether strong Gross spaces can exist when $\mathbf{b} > \omega_1$ holds. We prove here that strong Gross spaces are not destroyed by forcing with a finite support iteration in which each step amounts to forcing with the natural partial ordering P to adjoin a dominating function. Hence the iteration will be $\langle P_\alpha : \alpha \leq \kappa \rangle$ where P_α is the direct limit of $\langle P_\beta : \beta < \alpha \rangle$ if α is a limit and $P_{\alpha+1} \cong P_\alpha * \dot{Q}_\alpha$ where $\dot{Q}_\alpha$ denotes P defined in V^{P_α}.

The partial ordering $(P, \leq)$ is defined as follows. Let $IS = \{s \in \bigcup_{n\in\omega} {}^n\omega : s$ is strictly increasing$\}$. Then $P = \{(s,f) : s \in IS,\ f \in {}^\omega\omega,\ f$ strictly increasing$\}$, and we set $(s,f) \leq (t,g)$ iff $s \supseteq t$ and $\forall n f(n) \geq g(n)$ and $\forall n \geq$ length$(t)\, s(n) > g(n)$. It is clear that P and hence P_κ (by Theorem 6, §1) has the countable chain condition since any two elements of P with the same first coordinate are compatible. It is not difficult to see that forcing with P adds a function in ${}^\omega\omega$ which eventually dominates all functions in the ground model.

Hence, by what we claimed above, if we force with P_κ where $\mathrm{cf}(\kappa) > \omega_1$ over a model where strong Gross spaces exist, then we get a model where such spaces exist and $\mathbf{b} > \omega_1$ holds.

But we can get even more. Let V be the model in Theorem 1 where $\kappa > \omega_1$ is regular such that $\mathbf{c} = \kappa$ holds. Then strong Gross spaces of maximal dimension $\mathbf{c}$ exist in V. We claim that the continuum is not affected by forcing with P_κ. In fact, if $\dot{Q}$ denotes P defined in V^P then (since P satisfies the countable chain condition) $\Vdash_P |\dot{Q}| = \kappa$; furthermore $\kappa^\omega = \mathbf{c}^\omega = \mathbf{c} = \kappa$. Hence, by Theorem 7, §1 we have $|P * \dot{Q}| = \kappa$. By induction we get $|P_\kappa| = \kappa$, and hence by Theorem 8, §1 we conclude $\Vdash_{P_\kappa} \mathbf{c} = \kappa$. Clearly V^{P_κ} satisfies $\mathbf{b} = \kappa$ (κ is regular), and thus we get the following result:

THEOREM 2: *Suppose we force by adding κ Cohen reals such that in the extension V, κ is a regular cardinal and $\mathbf{c} = \kappa$ holds. Let P_κ be the finite support iteration of κ many Hechler forcing notions as defined in V.*

Then in the extension V^{P_κ} strong Gross spaces of (maximal) dimension **c** *exist.*

Proof: We have to show that strong Gross spaces are preserved under forcing with P_κ. So in V, let (E, Φ) be strongly Gross over k. Without loss of generality we may assume that $\dim E = \omega_1$. In fact, any uncountable dimensional subspace of E, if equipped with the restricted form, is strongly Gross, and conversely, if (E, Φ) is not strongly Gross in V^{P_κ} (since P_κ has the countable chain condition) then there exists an $\aleph_1$-dimensional subspace of E in V which is not strongly Gross in the extension.

In [B/D], a sequence $\langle a_\xi : \xi < \lambda\rangle$ in $[\omega]^\omega$ of length λ where $\mathrm{cf}(\lambda) > \omega$ is called **eventually narrow** iff $\forall a \in [\omega]^\omega \exists \xi < \lambda \forall \eta > \xi\ a - a_\eta$ is infinite. Theorem 3.3 [B/D] states that any eventually narrow sequence in V remains eventually narrow in V^{P_κ}. There is a close relation between preservation of eventually narrow sequences and preservation of strong Gross spaces. The following definition provides the link.

Definition: A sequence $\langle A_\xi : \xi < \lambda\rangle$ of subspaces $A_\xi \subseteq E$ with $\mathrm{cf}(\lambda) > \omega$ is called **eventually linearly narrow** iff for any infinite linearly independent set $A \subseteq E$ there exists $\xi < \lambda$ so that for every $\eta > \xi\ A - A_\eta$ is infinite. ∎

Now fix an enumeration $\langle x_\xi : \xi < \omega_1\rangle$ of E $[|E| = |k| \cdot \dim E = \aleph_1]$. We claim that $\langle (x_\xi)^\perp : \xi < \omega_1\rangle$ is eventually linearly narrow in V. [Otherwise, there exists an infinite linearly independent set $A \subseteq E$ such that $A - (x_\xi)^\perp$ is finite for uncountably many ξ. We may certainly find an infinite $A' \subseteq A$ and an unbounded $B \subseteq \omega_1$ such that $\forall \xi \in B A' \subseteq (x_\xi)^\perp$. By countability of k we conclude $\dim \mathrm{span} A'^\perp = \aleph_1$. This contradicts the assumption that E is strongly Gross in V.]

On the other hand, if in V^{P_κ} there exists an infinite dimensional subspace $U \subseteq E$ so that $\dim U^\perp = \aleph_1$ (i.e. E is not strongly Gross in V^{P_κ}), then clearly $\langle (x_\xi)^\perp : \xi < \omega_1\rangle$ is not eventually linearly narrow in V^{P_κ}.

From the previous two paragraphs we conclude that it suffices to show that eventually linearly narrow sequences are preserved by forcing with P_κ.

If the field k is finite we may apply Theorem 3.3 [B/D] without difficulty, since then $\langle (x_\xi)^\perp : \xi < \omega_1\rangle$ is even an eventually narrow sequence in the sense that $\forall A \in [E]^\omega \exists \xi < \omega_1 \forall \eta > \xi\ A - A_\xi$ is infinite. [Any $A \in [E]^\omega$ contains an infinite linearly independent set.]

If k is infinite this is clearly not true, since an arbitrary finite dimensional subspace (which is infinite as a set) is contained in uncountably many $(x_\xi)^\perp$. So we have to give new arguments.

As in the proof of Theorem 3.3 in [B/D] the crucial step is the successor step.

THEOREM 3: *Any eventually linearly narrow sequence $\langle A_\xi : \xi < \lambda\rangle$ in V remains eventually linearly narrow in V^P.*

Proof: By the way of contradiction assume that the theorem is not true. Then there exist $(s,f) \in P$ and a P-name $\dot{A}$ such that

$$(s,f) \Vdash \dot{A} \subset E \text{ is a countably infinite linearly independent set such that } \forall \xi < \lambda \exists \eta > \xi \dot{A} - A_\eta \text{ is finite.}$$

By the countable chain condition of P there exists a subspace $U \subset E$ of countable dimension, say $U = \bigoplus_{n\in\omega} ku_n$, such that $U \in V$ and

$$(s,f) \Vdash \dot{A} \subseteq U$$

We claim that there exists a P-name $\dot{H}$ so that

$$(1) \qquad (s,f) \Vdash \dot{H} \text{ enumerates } \dot{A} \text{ and } \forall i \in \omega\, \dot{H}(i) \notin \bigoplus_{j<i} ku_j$$

[In order to obtain such a name proceed as follows. In V, fix a well-ordering $\prec$ of U such that for all $x = \sum_{i=1}^m \alpha_i u_{m_i}$, $y = \sum_{i=1}^n \beta_i u_{n_i} \in U$ the following holds:

$$\max\{m_i : i \le m\} < \max\{n_i : i \le n\} \to x \prec y$$

It is clear that every infinite linearly independent family in U is given type ω by the ordering $\prec$ and its $\prec$-ith element does not lie in $\operatorname{span}\{u_j : j < i\}$. So let $\dot{H}$ be a P-name such that

$$(s,f) \Vdash \dot{H} \text{ enumerates } \dot{A} \text{ according to } \prec$$

(We use the existential completeness of forcing to get such $\dot{H}$.)]

Now for each $\xi < \lambda$ fix $(s_\xi, f_\xi) \le (s,f)$ and $n_\xi \in \omega$, if possible, such that

$$(2) \qquad (s_\xi, f_\xi) \Vdash \forall i \ge n_\xi\, \dot{H}(i) \in A_\xi$$

Clearly, (s_ξ, f_ξ) and n_ξ will be defined for ξ belonging to an unbounded subset of λ. By $\mathrm{cf}(\lambda) > \omega$ there exist u, n and an unbounded $B \subseteq \lambda$ such that $\forall \xi \in B\, s_\xi = u$ and $n_\xi = n$.

Again, by cf(λ) $> \omega$ we may choose $h \in {}^{\omega}\omega$ such that for all $m \in \omega$ the set $\{\xi \in B : h|m \subset f_\xi\}$ is cofinal in λ.

The rest of the proof of Theorem 3 is organized as follows. We will prove Lemma 2 which is an analog of Lemma 3.2 [B/D] under the assumption that Theorem 3 is false and then derive a contradiction. The proof will be by induction on elements of IS, justified by the following Lemma 1 which is Theorem 2.1 [B/D] and holds unconditionally.

Suppose D is an open and dense subset of the p.o. P. ("Open" means $\forall p, q \in P$ if $p \leq q$ and $q \in D$, then $p \in D$.) Define a sequence $\langle D_\alpha : \alpha < \omega_1\rangle$ of subsets of IS by induction as follows:

(1) Let $D_0 = \{s \in IS : \exists f\,(s, f) \in D\}$.

(2) Let $D_{\alpha+1} = \{s \in IS : (\exists n \geq |s|)(\forall k \in \omega)(\exists t \in D_\alpha)s \subseteq t$ and $|t| = n$ and $(\forall i)$ if $|s| \leq i < n$ then $t(i) > k\}$.

(3) If α is a limit ordinal then let $D_\alpha = \bigcup\{D_\beta : \beta < \alpha\}$.

Then $\langle D_\alpha : \alpha < \omega_1\rangle$ is a $\subseteq$-increasing sequence of subsets of IS. Since IS is countable there exists $\gamma < \omega_1$ such that $D_\gamma = D_{\gamma+1}$.

LEMMA 1 ([B/D], Theorem 2.1.): *Suppose $D \subseteq P$ is dense and open and $\langle D_\alpha : \alpha < \omega_1\rangle$ and γ are defined as above. Then $D_\gamma = IS$.*

Proof of Lemma 1: Suppose that for some $s \in IS$ we had $s \notin D_\gamma$. For $n < \omega$ let

$$W_n = \{t \in IS : s \subseteq t \text{ and } |t| = n \text{ and } t \in D_\gamma \text{ and } (\forall s \subset t' \subset t)t' \notin D_\gamma\}$$

It is not difficult to see that each W_n is finite. Hence we can define a strictly increasing $f \in {}^{\omega}\omega$ such that $(\forall n)f(n-1) > \max\{t(n-1) : t \in W_n\}$. Since D is dense we find $(t, g) \in D$ such that $(t, g) \leq (s, f)$. Now $t \in D_0$ and so some $t' \subseteq t$ belongs to some W_n where $n > |s|$. But this is impossible by the definition of f and since $t'(m) > f(m)$ for all $m \in \mathrm{dom}(t')$ with $m \geq |s|$. ∎

LEMMA 2: *For all $i \geq n$ and t such that $(t, h) \leq (u, h)$ the set*

$$Z_t(i) = \{F \subset U : F \text{ is a subspace of finite dimension and } \forall g \exists (t', g') \leq (t, g)$$
$$(t', g') \Vdash \dot{H}(i) \in F\}$$

is not empty.

Proof of Lemma 2: Let $i \geq n$. The set $D = \{(t,g) : \exists x\, (t,g) \Vdash \dot{H}(i) = x\}$ is dense and open. Hence by Lemma 1, $\bigcup_{\alpha<\omega_1} D_\alpha = IS$. The proof is by induction on α.

For $t \in D_0$ the statement is clear.

Let $t \in D_{\alpha+1} - D_\alpha$ such that $(t,h) \leq (u,h)$. There exist a sequence $(t_n)_{n\in\omega}$ in D_α and $m \in \omega$ such that

$$(\forall n)\, t \subseteq t_n \text{ and } |t_n| = m \text{ and } t_n(|t|) \geq n \text{ and } (t_n,h) \leq (t,h)$$

By induction hypothesis there exist $F_n \in Z_{t_n}(i)$. We may assume that all the F_n are of minimal dimension. Hence for every subspace $M \subseteq F_n$, $M \neq F_n$ there exists $g \in {}^\omega\omega$ so that

$$\forall (t',g') \leq (t_n,g)\, (t',g') \Vdash \dot{H}(i) \notin M$$

If $F = \sum_{n\in\omega} F_n$ is of finite dimension, then clearly $F \in Z_t(i)$ and we are done.

Suppose $\dim F = \infty$. We claim that this cannot happen. In V, we define a linearly independent set $\{y_l : l \in \omega\}$ in F by induction so that

$$\forall l \forall \xi \in B' := \{\xi \in B : h|m \subset f_\xi\}\, y_l \in A_\xi$$

This is a contradiction since $\langle A_\xi : \xi < \lambda\rangle$ then is not eventually linearly narrow in V.

We define $\{y_l : l \in \omega\}$ as follows:

1) We claim that there exists $y_0 \in F_0 - \{0\}$ so that $(\forall \xi \in B')\, y_0 \in A_\xi$. Suppose not. Then $F_0 \cap \bigcap_{\xi\in B'} A_\xi = \{0\}$ and hence there exist finitely many $\xi_1, \ldots, \xi_p \in B'$ so that

$$F_0 \cap \bigcap_{j=1}^{p} A_{\xi_j} = \{0\} \tag{3}$$

Choose $g \in {}^\omega\omega$ such that $\forall n \forall 1 \leq j \leq p\, g(n) \geq f_{\xi_j}(n)$. Since $F_0 \in Z_{t_0}(i)$ there exists $(t',g') \leq (t_0,g)$ such that

$$(t',g') \Vdash \dot{H}(i) \in F_0 \tag{4}$$

Then clearly for all $1 \leq j \leq p$ we have $(t',g') \leq (u, f_{\xi_j})$ and thus by (2)

$$(t',g') \Vdash \dot{H}(i) \in A_{\xi_j} \tag{5}$$

But (4) and (5) put together contradict (3).

2) Assume $y_1, \ldots, y_l$ have been chosen linearly independent such that for all $1 \leq j \leq l$ $y_j \in F_{n_j}$ and $\forall \xi \in B'$ $y_j \in A_\xi$.

There exists n_{l+1} so that $M := F_{n_{l+1}} \cap \sum_{j=1}^{l} F_{n_j} \subsetneqq F_{n_{l+1}}$. By minimality of $\dim F_{n_{l+1}}$ there exists $g \in {}^\omega\omega$ so that

$$\forall (t', g') \leq (t_{n_{l+1}}, g)\,(t', g') \Vdash \dot{H}(i) \notin M \tag{6}$$

Since $F_{n_{l+1}} \in Z_{t_{n_{l+1}}}(i)$ the set

$$N = \{x \in F_{n_{l+1}} : \exists (t', g') \leq (t_{n_{l+1}}, g)\,(t', g') \Vdash \dot{H}(i) = x\}$$

is not empty. By (6), $N \cap M = \emptyset$. Clearly dim span$N < \infty$.

Now choose $\xi_1, \ldots, \xi_p \in B'$ such that the dimension of span$N \cap A_{\xi_1} \cap \ldots \cap A_{\xi_p}$ is minimal. Then clearly we have $N \cap A_{\xi_1} \cap \ldots \cap A_{\xi_p} \subseteq$ span$N \cap A_{\xi_1} \cap \ldots \cap A_{\xi_p} \subseteq A_\xi$ for all $\xi \in B'$. Consequently, in case $N \cap A_{\xi_1} \cap \ldots \cap A_{\xi_p} \neq \emptyset$ we may choose y_{l+1} in this intersection arbitrarily. We claim that this case must occur.

Otherwise we have $N \cap A_{\xi_1} \cap \ldots \cap A_{\xi_p} = \emptyset$. Choose $g' \geq g, f_{\xi_1}, \ldots, f_{\xi_p}$. There exist $x \in N$ and $(t', g'') \leq (t_{n_{l+1}}, g')$ such that $(t', g'') \Vdash \dot{H}(i) = x$. But also $(t', g'') \leq (u, f_{\xi_j})$ and hence $(t', g'') \Vdash \dot{H}(i) \in A_{\xi_j}$ for all $1 \leq j \leq p$, and thus $x \in N \cap A_{\xi_1} \cap \ldots \cap A_{\xi_p}$, a contradiction.

Hence, Lemma 2 is proved. ∎

By Lemma 2 we may choose $F_i \in Z_u(i)$ for every $i \geq n$. Then by (1) there exists an increasing sequence $(i_l)_{l \in \omega}$ so that

$$(\forall l)\, F_{i_l} \cap \sum_{j<l} F_{i_j} \neq F_{i_l}$$

As in the proof of Lemma 2 we may find a linearly independent set $\{y_l : l \in \omega\}$ such that $y_l \in F_{i_l}$ and

$$\forall l \forall \xi \in B\, y_l \in A_\xi$$

Hence the sequence $\langle A_\xi : \xi < \lambda \rangle$ is not eventually linearly narrow in V, a contradiction. This proves Theorem 3. ∎

THEOREM 4: *Any eventually linearly narrow sequence remains eventually linearly narrow in V^{P_κ}.*

Proof: By induction on $\alpha \leq \kappa$. The successor step is handled by Theorem 3. For limit ordinals the proof of Theorem 3.3 [B/D] applies without difficulty. Suppose $\langle A_\xi, \xi < \lambda \rangle$ is eventually linearly narrow. If $\mathrm{cf}(\alpha) > \omega$ then P_α introduces no new subspaces of countable dimension that have not already appeared at some earlier step of the iteration. So we may assume $\mathrm{cf}(\alpha) = \omega$ and the Theorem is true for $\beta < \alpha$. Let $\langle \alpha_n, n \in \omega \rangle$ be an increasing sequence cofinal in α.

Suppose that the Theorem is false for α. So for some P_α-name $\dot{H}$ and some $p \in P_\alpha$ we have

$$p \Vdash \dot{H} \text{ enumerates a countably infinite linearly independent set in } E \text{ such that } \forall \xi < \lambda \exists \eta > \xi \; \mathrm{range}\dot{H} - A_\eta \text{ is finite.}$$

In V, for each $\xi < \lambda$ choose a condition $p_\xi \leq p$ and $n_\xi \in \omega$, if possible, so that

$$p_\xi \Vdash \forall i \geq n_\xi \, \dot{H}(i) \in A_\xi$$

Clearly, p_ξ and n_ξ will be defined for ξ belonging to some cofinal subset of λ. Since P_α is the directed limit of the P_β, $\beta < \alpha$, for each ξ we have $p_\xi \in P_{\alpha_m}$ for some $m \in \omega$. Choose n and m such that for cofinally many $\xi \in \lambda$ we have $p_\xi \in P_{\alpha_m}$ and $n_\xi = n$. Since P_{α_m} satisfies the countable chain condition there exists a P_{α_m}-generic filter G which contains cofinally many of the p_ξ's where $n_\xi = n$. Now define B as the set of ξ such that for some $p \in G$ $p \Vdash \forall i \geq n \dot{H}(i) \in A_\xi$. Clearly $B \in V[G]$ and B is unbounded in λ. Hence we conclude

$$V[G] \models \{\dot{H}[G](i) : i \geq n\} \subseteq \bigcap \{A_\xi : \xi \in B\}.$$

But this contradicts the inductive hypothesis that $\langle A_\xi, \xi < \lambda \rangle$ is eventually linearly narrow in $V^{P_{\alpha_m}}$. ∎

∎

Remark 1: The attentive reader may have noticed that in Theorem 2 we stated less than one could conceive to be true.

We forced twice. At first we adjoined Cohen reals to create strong Gross spaces over *arbitrary* fields. Then we adjoined Hechler reals and proved that the spaces existing over the fields in the first-step model are preserved. But a dominating function creates new fields as can be seen by arguing with infinitely many irreducible polynomials over the rationals (of unbounded degree). The dominating real gives a new subset of them (dominate the degrees). Look at the algebraic extension given by adjoining the zeros of the polynomials in it. We do not know whether Hechler Forcing produces strong Gross spaces so we do not know whether in the final extension strong Gross spaces exist over *all* fields.

To overcome this difficulty just adjoin κ Cohen reals (with finite support) not only at the beginning of the iteration but again and again after every dominating real. Then a new field occurs in an intermediate stage and, by Theorem 1, the next block of Cohen reals generates a strong Gross space of maximal dimension (which is κ) over that field. But this space could be killed again later in the iteration by a Cohen step. Fortunately this will not happen since every finite-support iteration of ccc partial orderings which preserve strong Gross spaces itself preserves such spaces. (This is just the limit step of the proof of Theorem 4, since there we used no additional information on the posets of the iteration.) But Cohen forcing is the finite-support limit of the trivial forcing with two incompatible elements. So we are done. I am indebted to Jim Baumgartner for this hint. ∎

Remark 2: We remark that Theorem 4 is true for vector spaces defined over arbitrary (not necessarily finite or countable) fields. ∎

Remark 3: Translating the proof of Theorem 3 back to the original situation in [B/D], one obtains a purely combinatorial argument avoiding elementary substructures. ∎

3. Nonexistence of strong Gross spaces is consistent with $\mathbf{p} = \omega_1$

By Theorem 1 in Chapter 2 of [B/Sp], strong Gross spaces do not exist if $\mathbf{p} > \omega_1$ holds. Since under $\mathbf{b} = \omega_1$ a strong Gross space always exists ([B/S], Ch. 4) it is natural to ask whether this is true under the weaker condition $\mathbf{p} = \omega_1$ ($\mathbf{p} \leq \mathbf{b}$ by Theorem 1a), §1). Here we give a negative answer to this question. We prove the following Theorem:

THEOREM 1: *Let V be a model of ZFC which has been obtained from a model for ZFC+GCH by adjoining many subsets of ω_1 by the standard σ-closed Cohen forcing. In V, let $\kappa > \omega_1$ be a regular cardinal such that $\kappa < 2^{\omega_1}$.*

Then in V there exists a ccc partial ordering P such that in the extension V^P there are no strong Gross spaces and $\mathbf{p} = \omega_1$ and $\mathbf{c} = \kappa$ hold.

Proof: The first step in the proof will be to define a partial ordering P^k where k is an arbitrary finite or countable field such that no quadratic space over k in the ground model will be strongly Gross in the extension V^{P^k}.

Elements of P^k are pairs (s, A) where

1) $s = (s_1, \ldots, s_n)$ is a finite sequence of functions $s_i : \omega \to k$ such that $(\forall i)$ $\text{support}(s_i) = \{m : s_i(m) \neq 0\}$ is finite. (The s_i's will describe vectors.)

2) A is a finite set of functions $f : \omega \to k$. (The f's may be thought of as sequences of Fourier coefficients.)

On P^k we define an ordering $\leq$ by setting

$$(s, A) \leq (t, B) \text{ iff } s \supseteq t \text{ and } A \supseteq B \text{ and}$$
$$(\forall i) \text{ if } i \geq \text{length}(t) \text{ then } (\forall f \in B) \textstyle\sum_{n \in \omega} s_i(n) \cdot f(n) = 0$$

In the sequel, we will abbreviate the formula $\sum_{n\in\omega} s_i(n) \cdot f(n) = 0$ by writing $s_i \perp f$. Needless to say that the dot in this formula is the multiplication in the field k.

Since two conditions with the same first coordinate are compatible and $|k| \leq \aleph_0$, P^k clearly has the countable chain condition (is even σ-centered).

Now let G be P^k-generic over V and set $S = \bigcup \text{pr}_1 G$.

Proposition 1: *S is an infinite sequence of vectors S_i, and if $f : \omega \to k$ is in V, then there exists $i < \omega$ such that $(\forall j \geq i) S_j \perp f$.*

Proof of Prop. 1: It suffices to see that for arbitrary n and f the set of pairs (s, A) such that $|s| \geq n$ and $f \in A$ is dense in P^k. Let t be of length $\geq n$ such that $t \supseteq s$ and t_i is the zero vector for $i \geq |s|$. Then clearly $(t, A \cup \{f\})$ is as desired and extends (s, A). ∎

The intention is that in every quadratic space (E, Φ) in V the vectors given by S will describe subspaces with a large orthogonal complement. So we must be sure that they are of infinite dimension. This is accomplished by the following Proposition.

Proposition 2: *Considered as vectors in the k-vector space k^ω, the S_i's span an infinite dimensional subspace.*

Proof of Prop. 2: By genericity, it suffices to show that for every n the set

$$D_n = \{(s, A) \in P^k : (\exists i < |s|)(\exists m \geq n)\, s_i(m) \neq 0\}$$

is dense.

Let $(s, A) \in P^k$ where $A = \{f_1, \ldots, f_p\}$. Consider the following system of p homogeneous equations in $p+1$ unknowns $\xi_n, \ldots, \xi_{n+p}$

$$\begin{array}{ccccccc} f_1(n)\xi_n & + & \ldots & + & f_1(n+p)\xi_{n+p} & = & 0 \\ \vdots & & & & \vdots & & \vdots \\ f_p(n)\xi_n & + & \ldots & + & f_p(n+p)\xi_{n+p} & = & 0 \end{array}$$

and let $(\xi_n, \ldots, \xi_{n+p})$ be a nontrivial solution vector. We set

$$t_{|s|}(m) = \begin{cases} \xi_m & \text{if } n \le m \le n+p \\ 0 & \text{otherwise} \end{cases}$$

and $t = s^\frown t_{|s|}$. Then clearly $(t, A) \le (s, A)$ and $(t, A) \in D_n$. ∎

Now we are able to prove the following theorem:

THEOREM 2: *In V, let (E, Φ) be a quadratic space over k of uncountable dimension. If $U \subseteq E$ is a subspace of infinite dimension and $X \subseteq E$ is uncountable with $\mathrm{cf}(|X|) > \omega$ and $U, X \in V$, then in V^{P^k} there exists an infinite dimensional $U' \subseteq U$ such that $U'^{\perp} \cap X$ has cardinality $|X|$. Consequently, $\dim U'^{\perp} \ge |X|$ and (E, Φ) is not strongly Gross in V^{P^k}.*

Proof of Theorem 2: Let $U = \bigoplus_{n \in \omega} k u_n$. For every $x \in X$ set $f_x = \langle \Phi(x, u_n) : n \in \omega \rangle$. Then clearly $f_x \in V$. Hence by Proposition 1 there exists $i_x \in \omega$ such that $(\forall i \ge i_x) S_i \perp f_x$. We conclude $\forall i \ge i_x$

$$\Phi(x, \sum_{n \in \omega} S_i(n) u_n) = \sum_{n \in \omega} S_i(n) f_x(n) = 0$$

By $\mathrm{cf}(|X|) > \omega$ we may choose i and $X' \subseteq X$ such that $|X'| = |X|$ and $(\forall x \in X') i_x = i$. Hence, if we let

$$U' = \text{ span } \{\sum_{n \in \omega} S_j(n) u_n : j \ge i\}$$

we have $X' \subseteq U'^{\perp}$ and $\dim U' = \aleph_0$ by Proposition 2. ∎

Theorem 2 does not rule out the possibility that in V^{P^k} either strong Gross spaces in V over a field distinct from k are preserved or spaces are strongly Gross which have not been in V. In order to get a model where provably no strong Gross space exists we iterate forcing with P^k where again and again k runs through all possible fields. Clearly it suffices only to consider fields with a subset of ω as underlying set.

Our ground model V satisfies CH and $2^{\omega_1} > \kappa > \omega_1$. We will make sure that in the final extension $\mathbf{c} = \kappa < 2^{\omega_1}$ will hold. Then by Theorem 1c), §1 we know that $\mathbf{p} = \omega_1$ is true.

By induction we define a finite support iteration $\langle P_\alpha : \alpha \leq \kappa \rangle$ such that the following are true:

1) For all $\alpha < \kappa$ we have $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ such that

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \text{ has the countable chain condition and } |\dot{Q}_\alpha| = \kappa$$

2) If G_κ is P_κ-generic over V and k is a finite or countable field in $V[G_\kappa]$ with a subset of ω as its underlying set, then for arbitrarily large $\alpha < \kappa$ we have $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ where $\dot{Q}_\alpha$ denotes P^k defined in $V[G_\alpha]$. (It goes without saying that by Theorem 5, §1 $k \in V[G_\alpha]$ for some $\alpha < \kappa$.)

Suppose $\alpha < \kappa$ and P_β and $\dot{Q}_\beta$ have been determined such that 1) holds for all $\beta < \alpha$. By Theorem 6, §1, P_β has the countable chain condition for all $\beta \leq \alpha$. So, using induction and Theorem 7, §1 we conclude $|P_\alpha| \leq \kappa$. Putting together these facts, by Theorem 8, §1 we get $\Vdash_{P_\alpha} \mathbf{c} \leq \kappa$.

Hence there is a P_α-name $\langle \dot{k}_\alpha^\gamma : \gamma < \kappa \rangle$ such that

$$\Vdash_{P_\alpha} \langle \dot{k}_\alpha^\gamma : \gamma < \kappa \rangle \text{ enumerates all finite or countable fields with a subset of } \omega \text{ as underlying set}$$

Fix $\pi : \kappa \to \kappa \times \kappa$ such that for every $(\beta, \gamma) \in \kappa \times \kappa$ there are arbitrarily large $\alpha < \kappa$ such that $\pi(\alpha) = (\beta, \gamma)$, and whenever $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$.

Let $\alpha = \pi(\beta, \gamma)$. Since $\beta \leq \alpha$ we may treat $\dot{k}_\beta^\gamma$ as a P_α-name and hence we may define $\dot{Q}$ as a P_α-name such that

$$\Vdash_{P_\alpha} \dot{Q} = P^{\dot{k}_\beta^\gamma}$$

Then clearly, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ has the countable chain condition", and by $\Vdash_{P_\alpha} \mathbf{c} \leq \kappa$ as seen above we also have $\Vdash_{P_\alpha} |\dot{Q}_\alpha| \leq \kappa$. This completes the definition of $\langle P_\alpha : \alpha \leq \kappa \rangle$.

Now we are able to finish the proof of Theorem 1. As in Theorem 2, we can get even more:

THEOREM 1 (revised): *Let G_κ be P_κ-generic over V. In $V[G_\kappa]$, let k be a finite or countable field and (E, Φ) a quadratic space over k of uncountable dimension.*

Then for every subspace $U \subsetneq E$ of infinite dimension and every uncountable $X \subseteq E$ with $|X| < \kappa$ and $\mathrm{cf}(|X|) > \omega$ there exists a subspace $U' \subsetneq U$ of infinite dimension such that $U'^{\perp} \cap X$ has cardinality $|X|$ and hence $\dim(U'^{\perp}) \geq |X|$.

In particular, (E, Φ) is not strongly Gross.

Furthermore, $V[G_\kappa]$ satisfies $\mathbf{p} = \omega_1$.

Proof of revised Theorem 1: We may certainly assume that the underlying set of k is a subset of ω and $\dim U = \aleph_0$.

Set $E' = \mathrm{span}(X \cup U)$, $\Phi' = \Phi | E' \times E'$. By Theorem 5, §1 there exists $\beta < \kappa$ such that k, U, X and (E', Φ') are all in $V[G_\beta]$. Then k must be the denotation of some $\dot{k}^{\gamma}_{\beta}$. Let $\pi(\alpha) = (\beta, \gamma)$. By construction, $\dot{Q}_\alpha$ denotes P^k defined in $V[G_\alpha]$; hence $V[G_{\alpha+1}] = V[G_\alpha]^{P^k}$. By Theorem 2, in $V[G_{\alpha+1}]$ there exists a subspace $U' \subsetneq U$ of infinite dimension such that $U'^{\perp} \cap X$ has cardinality $|X|$.

As for the second statement we have already seen that $\Vdash_{P_\kappa} \mathbf{c} \leq \kappa$. It is obvious that during the iteration cofinally many times new subsets of ω get added. Hence we conclude $\Vdash_{P_\kappa} \mathbf{c} = \kappa$. Since P_κ is ccc and thus preserves cardinals, $2^{\omega_1} > \kappa$ holds in $V[G_\kappa]$ since it holds in V. Hence by Theorem 1c), §1, $\mathbf{p} = \omega_1$ holds in $V[G_\kappa]$. This completes the proof of Theorem 1. ∎

Remark 1: Since under the assumption $\mathbf{b} = \omega_1$ a strong Gross space always exists, the model V^{P_κ} of Theorem 1 clearly satisfies $\mathbf{b} > \omega_1$. It even satisfies $\mathbf{b} = \mathbf{c}$, since by the following fact, at every step of the iteration where we force with P^k such that k is infinite a dominating function is added.

FACT: *Suppose k is a countably infinite field. Then forcing with P^k adjoins a function $f : \omega \to \omega$ which dominates all functions in ${}^\omega\omega$ in the ground model.*

Proof: In V, fix a bijection $\pi : \{t \in k^\omega : \{n : t(n) \neq 0\} \text{ finite }\} \to \omega$. Let G be P^k-generic over V and let t_i, $i \in \omega$, enumerate $S = \bigcup \mathrm{pr}_1 G$ such that $\forall i \exists j \geq i\, t_i(j) \neq 0$. Define $f(n) = \pi(t_n)$. We claim that f dominates the ground model reals. Given $g \in ({}^\omega\omega)^V$ define $h_g \in k^\omega$ as follows. By induction on n choose $h_g(n)$ so that $\forall t \in \pi^{-1}\{0, \ldots, g(n)\}$ if $n = \max\{i : t(i) \neq 0\}$ then $t \not\perp h_g$. Since k is infinite such a choice is possible. Clearly $h_g \in V$.

If now (s, A) is in G and $h_g \in A$ then $\forall t_i \in S - s$ we must have $f(i) = \pi(t_i) > g(i)$ so f dominates g. ∎

Remark 2: In the model of Theorem 1, $2^\omega < 2^{\omega_1}$ holds. Once we have obtained such a model, it is possible to get a model where no strong Gross spaces exist and $\mathbf{p} = \omega_1$ but $2^\omega = 2^{\omega_1}$ holds. This kind of argument we heard from Jim Baumgartner.

In fact the model of Theorem 1 is a two-step model. At first we force with the Cohen algebra over a model of GCH to get $2^{\omega_1} > \kappa$ and then via P_κ to make sure that there are no strong Gross spaces. In the final extension there is a witness W to $\mathbf{p} = \omega_1$ and that witness has cardinality ω_1. Of course P_κ has cardinality κ. Thus if we look at the model $V[W, P_\kappa][G_\kappa]$, where G_κ is the P_κ-generic set, it is clear that this is a submodel of the original two-step model so there are no strong Gross spaces in it. It also contains W so it must be true that $\mathbf{p} = \omega_1$. Finally, adding W and P_κ required no more than κ subsets of ω_1 so we must have $2^{\omega_1} = 2^\omega = \kappa$, as desired. ∎

References

[B] J.E. Baumgartner, *Iterated forcing*, in *Surveys of Set Theory* (A.R.D. Mathias, ed.), London Mathematical Society Lecture Note Series, no. 87, Cambridge University Press, Cambridge, 1983, pp. 1–59.

[B/D] J.E. Baumgartner and P. Dordal, *Adjoining dominating functions*, J. Symbolic Logic **50** (1985), 94–101.

[B/Sp] J.E. Baumgartner and O. Spinas, *Independence and consistency proofs in quadratic form theory*, J. Symbolic Logic **56** (1991), 1195–1211.

[Ba/G] W. Baur and H. Gross, *Strange inner product spaces*, Comment. Math. Helv. **52** (1977), 491–495.

[vD] E.K. van Douwen, *The integers and topology*, in *Handbook of Set-theoretic Topology* (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 111–167.

[G] H. Gross, *Quadratic forms in infinite dimensional vector spaces*, Progress in Mathematics, Vol. 1, Birkhäuser, Boston, 1979.

[G/O] H. Gross and E. Ogg, *Quadratic spaces with few isometries*, Comment. Math. Helv. **48** (1973), 511-519.

[J] T. Jech, *Multiple forcing*, Cambridge Tracts in Mathematics, Vol. 88, Cambridge University Press, Cambridge, 1986.

[K] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.

[Ma/Ma] F. Maeda and S. Maeda, *Theory of Symmetric Lattices,* Springer-Verlag, Berlin, 1970.

[Sh/Sp] S. Shelah and O. Spinas, *How large orthogonal complements are there in a quadratic space?,* in preparation.

[Sp1] O. Spinas, *Konsistenz- und Unabhängigkeitsresultate in der Theorie der quadratischen Formen,* Ph.D. Thesis, University of Zürich, 1989.

[Sp2] O. Spinas, *An undecidability result in lattice theory,* Abstracts of papers presented to the AMS, vol.11, no.2. p.161, March 1990.

[Sp3] O. Spinas, *Independence results on AC-lattices,* in preparation.

[Sp4] O. Spinas, *Cardinal invariants and quadratic forms,* in *Set Theory of the Reals,* Proceedings of the Bar Ilan Conference in honour of Abraham Fraenkel, 1991, to appear.